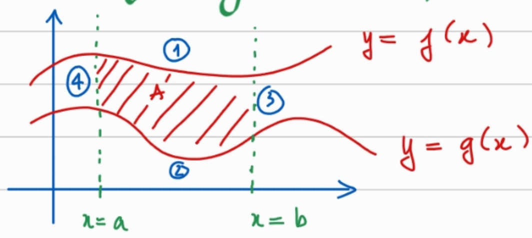
June 3rd, 2024

Lecture 8

Areas between curves

1) Integrating with respect to x



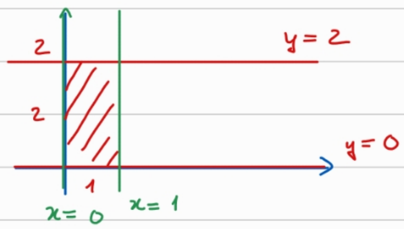
Problem: Compute the area of domain A, which is bounded by two curves $y = f(x)$, $y = g(x)$ and $x = a$, $x = b$.?

Theorem The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$, $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Example: Area between

$$\begin{cases} y = 2 \\ y = 0 \end{cases} \qquad \begin{cases} x = 0 \\ x = 1. \end{cases}$$

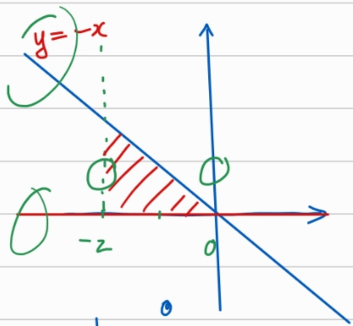


$$\text{Area} = 1.2. = \int_0^1 |2 - 0|\, dx = \int_0^1 2\,dx$$

$$= 2x\Big|_0^1 = 2.$$

Example: Area between

$$\begin{cases} y = -x \\ y = 0 \end{cases} \qquad \begin{cases} x = -2 \\ x = 0. \end{cases}$$



$$\text{Area} = \int_{-2}^{0} |-x - 0|\, dx = \int_{-2}^{0} |-x|\, dx$$

there $|-x| = -x$ since $x \in [-2, 0]$

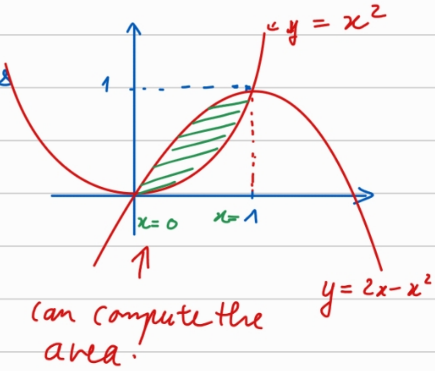
$$\text{Area} = \int_{-2}^{0} (-x)\, dx = -\int_{-2}^{0} x\, dx = 2.$$

$$\int_{-2}^{0} x\, dx = \frac{x^2}{2}\Big|_{-2}^{0} = \frac{0}{2} - \left(\frac{(-2)^2}{2}\right) = -2.$$

Area is negative.

Example 3. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

1) Find intersections of two curves to find $x = a$, $x = b$.
2) Compute the area.



1) Find intersections.

Let two equations equal and solve for x.

Solve $\quad f(x) = g(x)$

i.e. $\quad x^2 = 2x - x^2$

$\Rightarrow \quad 2x^2 - 2x = 0$

$\Rightarrow \quad x(x-1) = 0$

$\Rightarrow \quad \left[\begin{array}{l} x = 0 \\ x = 1 \end{array}\right.$.

2) Compute the area:

$$\int_0^1 |x^2 - (2x - x^2)|\, dx = \int_0^1 |2x^2 - 2x|\, dx \qquad (1)$$

$$|u| = \begin{cases} u & \text{if } u \geq 0 \\ -u & \text{if } u < 0. \end{cases}$$

Simplest way is draw a table. Solve for $u = 0$, put values to the table.

x	$-\infty$	0	1/2	1	$+\infty$
$u = 2x^2 - 2x$			$-$		

exclude since we only care about the region between $x = 0$, $x = 1$.

$2 \cdot \frac{1}{2^2} - 2 \cdot \frac{1}{2} = \frac{1}{2} - 1 < 0$

Thus integral (1) becomes:

$$-\int_0^1 (2x^2 - 2x)\, dx.$$

$$= -\left(2\frac{x^3}{3} - x^2\right)\Big|_0^1$$

$$= -\left(\frac{2}{3}1^3 - 1^2\right)$$

$$= -\left(-\frac{1}{3}\right)$$

$$= \frac{1}{3}.$$

Example 4. Find the approximate area of the region bounded by curves

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$$y = \frac{x}{\sqrt{x^2+1}}, \quad y = x^4 - x.$$

1) Solve for intersection points.

Solve. $\frac{x}{\sqrt{x^2+1}} = x^4 - x.$, $x = 0$, $x \approx 1.18$.

2) Area:

$$A = \int_0^{1.18} \left| \frac{x}{\sqrt{x^2+1}} - (x^4 - x) \right| dx.$$

$$= \int_0^{1.18} \left(\underbrace{\frac{x}{\sqrt{x^2+1}} - (x^4 - x)}_{>0} \right) dx.$$

?

How to compute this integral ?

$$= \int_0^{1.18} \frac{x}{\sqrt{x^2+1}} dx - \underbrace{\int_0^{1.18} (x^4 - x)\, dx}_{\text{easy.}}$$

Substitution.

$\rightarrow u = x^2 + 1 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$$\begin{cases} x = 1.18 \\ x = 0 \end{cases} \Rightarrow \begin{cases} u = 1.18^2 + 1 = 2.39. \\ u = 1 \end{cases}$$

$$\int_1^{2.39} \frac{1}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2} \int_1^{2.39} u^{-1/2}\, du$$

$$= \frac{1}{2} \left. \frac{u^{1/2}}{1/2} \right|_1^{2.39} = \left. u^{1/2} \right|_1^{2.39}. = \sqrt{2.39} - 1.$$

Example 5. Find the area of the region between curves

$$\begin{cases} y = \sin x \\ y = \cos x \end{cases} \quad \begin{cases} x = 0 \\ x = \pi/2 \end{cases}$$

$$\text{Area} = \int_0^{\pi/2} |\sin x - \cos x|\, dx. \qquad \underbrace{\int_0^{\pi/2} (\sin x - \cos x)\, dx}_{\text{not correct}}$$

We have to solve solution of $\{ \sin x = \cos x$ $\Rightarrow$ determine the value of $|\sin x - \cos x|$.

Example 5. Find the area of the region between curves

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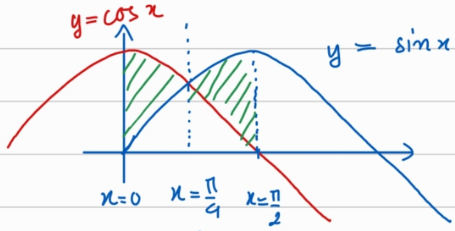
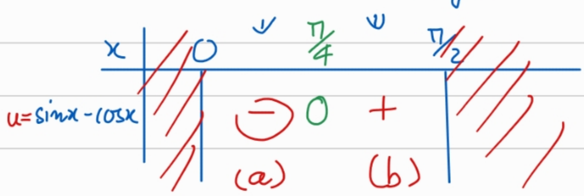
We have to solve solution of $\sin x = \cos x$ $\Rightarrow$ determine the value of $|\sin x - \cos x|$.

(Intersections of red curve with the blue curve)

Solve for x: $\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$ (because we are considering the region $x \in [0, \frac{\pi}{2}]$)





$$\text{Area} = \int_0^{\pi/2} |\sin x - \cos x|\, dx.$$

$$= \int_0^{\pi/4} |\underbrace{\sin x - \cos x}_{u}|\, dx + \int_{\pi/4}^{\pi/2} |\sin x - \cos x|\, dx$$

(a) $u = \sin x - \cos x$. (b)

$$= \int_0^{\pi/4} |u|\, dx + \int_{\pi/4}^{\pi/2} |u|\, dx$$

$u < 0$ $\qquad$ $u > 0$

$$= \int_0^{\pi/4} (-u)\, dx + \int_{\pi/4}^{\pi/2} u\, dx$$

$$= -\int_0^{\pi/4} (\sin x - \cos x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx.$$

$$= -\left(-\cos x - \sin x\right)\Big|_0^{\pi/4} + \left(-\cos x - \sin x\right)\Big|_{\pi/4}^{\pi/2}$$

$$= -\left[\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) - (-1 - 0)\right] + \left[(-0 - 1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right]$$

$$= -\left[-\sqrt{2} + 1\right] + \left[-1 + \sqrt{2}\right]$$

$$= \sqrt{2} - 1 \quad - 1 + \sqrt{2}$$

$$= 2\sqrt{2} - 2 \qquad \boxtimes$$

2) Integrating with respect to y.

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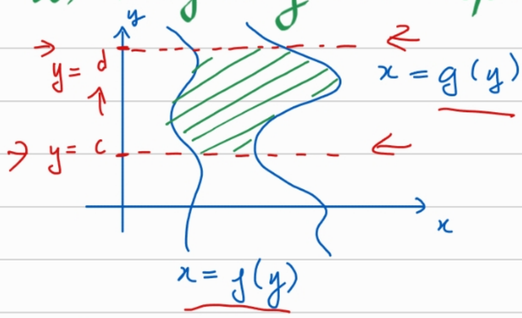
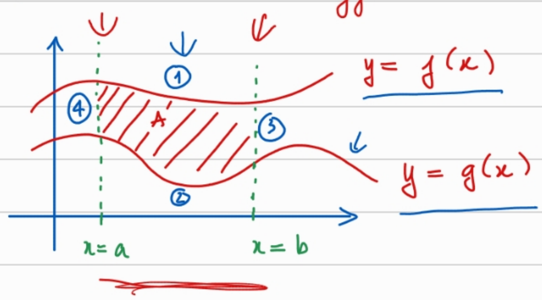


Problem: Find the area of the region bounded by curves $x = f(y)$, $x = g(y)$, between $y = c$, $y = d$.

Theorem. The area is

$$A = \int_c^d |f(y) - g(y)|\, dy.$$

Note that this is different with the above problem (variable x)



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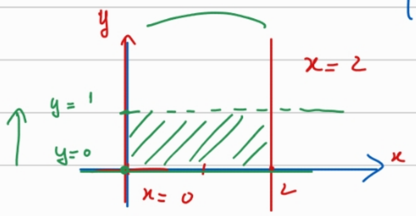
$$A = \int_a^b |f(x) - g(x)|\, dx$$

Example 1. Area of $\begin{cases} x = 2 & (1) \\ x = 0 & (2) \end{cases}$ $\begin{cases} y = 0 \\ y = 1. \end{cases}$



$$\text{Area} = 1.2 = \int_0^1 |2 - 0|\, dx$$

$$= \int_0^1 2\, dx = 2.x\Big|_0^1 = 2.$$

Example 2.

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

We have $y = x - 1$ $\quad f(x)$ okay.

$y^2 = 2x + 6 \Rightarrow$ cannot have $y = g(x)$. not okay.

$\Rightarrow$ we cannot integrating with variable x.

But for variable y:

$$\begin{cases} y = x - 1 \\ y^2 = 2x + 6 \end{cases} \Rightarrow \begin{cases} x = y + 1 \\ 2x = y^2 - 6 \end{cases} \Rightarrow \begin{cases} x = y + 1 = f(y) \\ x = \frac{1}{2}y^2 - 3 = g(y) \end{cases}$$

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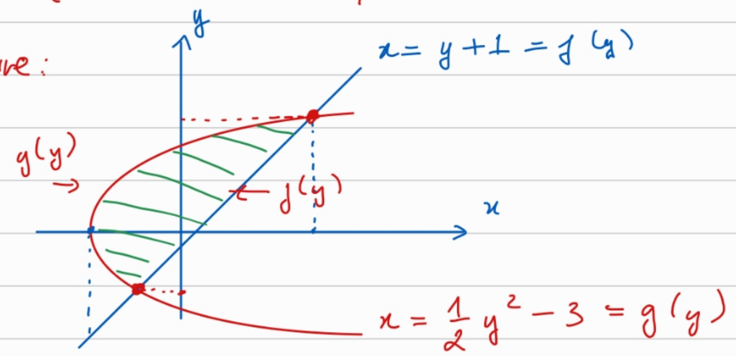
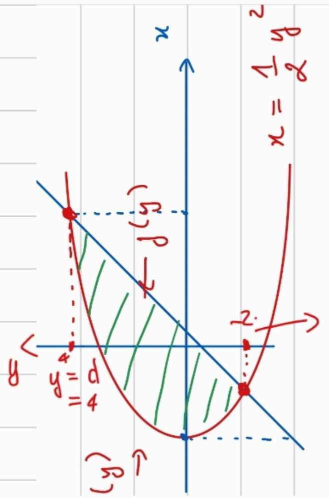
Picture:





1) Solve for intersection points of two curves.

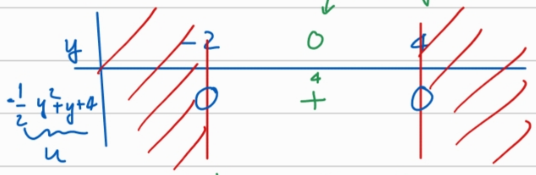
$$f(y) = g(y)$$
$$\Rightarrow y + 1 = \frac{1}{2}y^2 - 3$$
$$\Rightarrow 2y + 2 = y^2 - 6$$
$$\Rightarrow y^2 - 2y - 8 = 0$$
$$\Rightarrow (y+2)(y-4) = 0$$
$$\Rightarrow \left[\begin{array}{l} y = -2 = c \\ y = 4 = d. \end{array}\right.$$

$$\text{Area} = \int_c^d |f(y) - g(y)|\, dy = \int_{-2}^{4} |(y+1) - (\tfrac{1}{2}y^2 - 3)|\, dy.$$

$$= \int_{-2}^{4} |\underline{-\tfrac{1}{2}y^2 + y} + 4|\, dy.$$

$$|u| = \begin{cases} u & u \geq 0 \\ -u & u < 0 \end{cases}$$

pick $y = 0$



we have $u = -\frac{1}{2}y^2 + y + 4 > 0$.

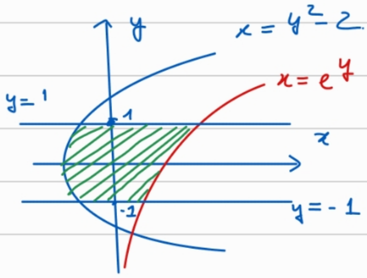
$$\text{Area} = \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4\right) dy. \text{ (keep the sign)}$$

$$= \left(-\frac{1}{2}\frac{y^3}{3} + \frac{y^2}{2} + 4y\right)\Big|_{-2}^{4}$$

$$= 18.$$

Ex: Find the area of the region enclosed by

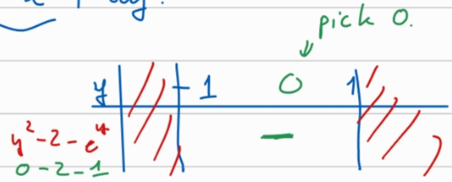
$$\begin{cases} x = y^2 - 2 = f(y) & y = 1 \\ x = e^y = g(y) & y = -1 \end{cases}$$



$$\text{Area} = \int_c^d |f(y) - g(y)|\, dy$$

$$= \int_{-1}^{1} |(y^2 - 2) - e^y|\, dy.$$

$$|u| = \begin{cases} u & u \geq 0 \\ -u & u < 0. \end{cases}$$



Solve for $y^2 - 2 - e^y = 0$ no solution

$$\text{Area} = -\int_{-1}^{1} u\, dy = -\int_{-1}^{1} (y^2 - 2 - e^y)\, dy$$

$$= -\left(\frac{y^3}{3} - 2y - e^y\right)\Big|_{-1}^{1}$$

$$= -\left[\left(\frac{1}{3} - 2 - e\right) - \left(\frac{-1}{3} + 2 - e^{-1}\right)\right]$$

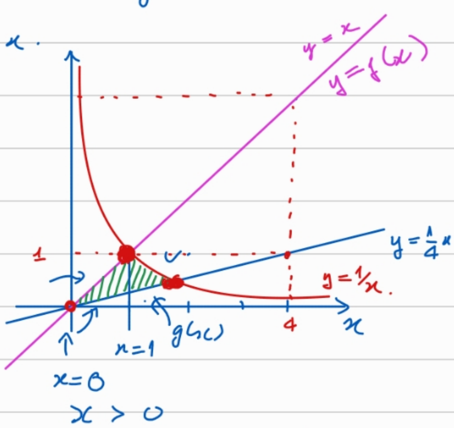
$$= -\left[\underbrace{\frac{2}{3} - 4}_{-10/3} - e + e^{-1}\right]$$

$$= \frac{10}{3} + e - e^{-1}$$ ☐

Example. Find the area of the region enclosed by the curves

$y = \frac{1}{x}$, $y = x$, $y = \frac{1}{4}x$.

using both methods
a) integrating with x
b) integrating with y.



a) variable x.

find $\begin{cases} y = f(x) \\ y = g(x) \end{cases}$

Observe that we can not have a unique function $y = f(x)$ on the whole region.

$\Rightarrow$ We have to split the region into smaller subregions such that in each region, we have unique $y = f(x)$

To do that, we solve for the intersection points of curves

3 curves $y = f(x) = x$ (pink) 1) $f(x) = g(x)$ 2) $f(x) = h(x)$ 3) $g(x) = h(x)$

3 curves $y = f(x) = x$ (pink)
$y = g(x) = \frac{1}{4}x$ (blue)
$y = h(x) = \frac{1}{x}$ (red)

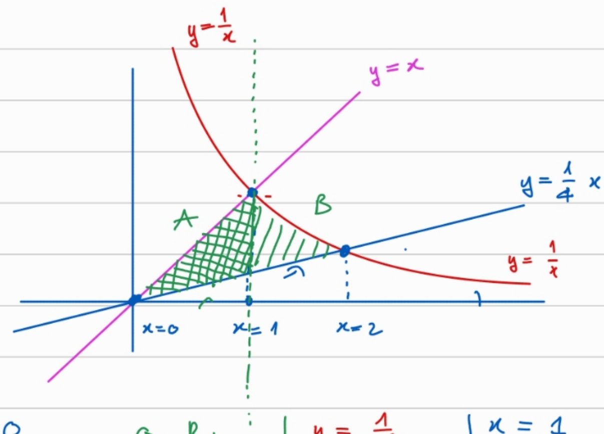
1) $f(x) = g(x)$ → intersection of pink curve and blue curve
2) $f(x) = h(x)$ → intersection of pink curve with red curve
3) $g(x) = h(x)$ → intersection of blue with red curve.

1) $f(x) = g(x) \Rightarrow x = \frac{1}{4}x \Rightarrow 4x = x \Rightarrow 3x = 0 \Rightarrow x = 0$

2) $x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow \left[\begin{array}{l} x = 1 \\ x = -1 \end{array}\right. \Rightarrow x = 1$
$(x > 0)$

3) $\frac{1}{4}x = \frac{1}{x} \Rightarrow \frac{1}{4}x^2 = 1 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$
$\Rightarrow x = 2$ $(x > 0)$.

Three intersection points
⇒ divide the region int two smaller regions:
A : $x = 0$ to $x = 1$
B : $x = 1$ to $x = 2$.
such that on each region we have unique function "$y = f(x)$"



On A : $\begin{cases} y = x \\ y = \frac{1}{4}x \end{cases}$ $\begin{cases} x = 0 \\ x = 1 \end{cases}$

On B : $\begin{cases} y = \frac{1}{x} \\ y = \frac{1}{4}x. \end{cases}$ $\begin{cases} x = 1 \\ x = 2. \end{cases}$

Now compute the areas of A and B → combine together.

Area of A: $\int_0^1 \left|x - \frac{1}{4}x\right| dx = \int_0^1 \frac{3}{4}x\,dx = \frac{3}{4}\frac{x^2}{2}\Big|_0^1 = \frac{3}{8}$

Area of B: $\int_1^2 \left|\frac{1}{x} - \frac{1}{4}x\right| dx = \left(\ln|x| - \frac{x^2}{8}\right)\Big|_1^2$

$= \left[\left(\ln 2 - \frac{1}{2}\right) - \left(0 - \frac{1}{8}\right)\right] = \ln 2 - \frac{3}{8}.$

Area of the original region
$= A + B$
$= \frac{3}{8} + \left(\ln 2 - \frac{3}{8}\right)$
$= \ln 2$